ISRAEL JOURNAL OF MATHEMATICS, Vol 45, No 1, 1983

STRUCTURE OF SPACES OF C^∞-FUNCTIONS ON NUCLEAR SPACES

BY

J. F. COLOMBEAU AND O. T. W. PAQUES[†]

ABSTRACT

Let E be a real nuclear locally convex space; we prove that the space $\mathscr{E}_{ub}(E)$, of all C^∞-functions of uniform bounded type on E, coincides with the inductive limit of the spaces $\mathscr{E}_{Nbc}(E_V)$ (introduced by Nachbin–Dineen), when V ranges over a basis of convex balanced 0-neighbourhoods in E. Let E be a real nuclear bornological vector space; we prove that the space $\mathscr{E}(E)$ of all C^∞-functions on E coincides with the projective limit of the spaces $\mathscr{E}_{Nbc}(E_B)$, when B is a closed convex balanced bounded subset of E. As a consequence we obtain some density results and a version of the Paley–Wiener–Schwartz theorem.

Introduction

Recent clarifications of differential calculus in locally convex spaces in Colombeau [4] and new applications of the spaces of C^∞-functions on nuclear spaces in Colombeau [5], [6], motivate a deeper study of these spaces of C^∞-functions. The main spaces of C^∞-functions (due to their mathematical properties and their relevance in applications) are the space $\mathscr{E}(E)$ of all C^∞-functions over a real nuclear bornological vector space E and the space $\mathscr{E}_{ub}(E)$ of all C^∞-functions of uniform bounded type over a real nuclear locally convex space E (this space $\mathscr{E}_{ub}(E)$ was introduced more recently in Colombeau–Mujica [9], Colombeau–Paques [10], Colombeau [4]).

In this paper we prove that these two spaces $\mathscr{E}(E)$ and $\mathscr{E}_{ub}(E)$ may be considered respectively as projective and inductive limits of spaces $\mathscr{E}_{Nbc}(H)$, which are "very good" spaces of C^∞-functions on separable real Hilbert spaces H introduced in Nachbin–Dineen [13]. In the complex case, i.e. for holomorphic functions over E, similar results had previously been proved in Colombeau–Matos [7], [8], but the proofs in the real case are quite different and

[†] Research done during the stay of this author at the University of Bordeaux (France) in the academic year 1980–1981.

Received May 20, 1982

more difficult. These results bring important clarifications of these concepts and some consequences (density results and Paley–Wiener–Schwartz theorems) are explained at the end of the paper.

I. Recalls, notations and terminology

We use classical notations and terminology (see Colombeau [4], Ansemil–Colombeau [2], Abuabara [1], Colombeau–Matos [7], [8], Colombeau–Mujica [9], Gupta [11] and Nachbin–Dineen [13]). If E is a real locally convex space (l.c.s., for short) we recall that $\mathscr{E}_{ub}(E)$ is the inductive limit, when V ranges over a base of convex balanced 0-neighbourhoods of E, of the spaces $\mathscr{E}_b(E_V)$ (the space of all infinitely differentiable functions on E_V which are bounded, with all their derivatives on each bounded subset of E_V), i.e. an element f of $\mathscr{E}_{ub}(E)$ may be considered as a function on E that may be factorized as $f = \tilde{f} \circ s_V$, for some V, where $s_V : E \to E_V$ denotes the canonical map ($E_V = E/p_V^{-1}(0)$ normed by the gauge p_V of V) and with $\tilde{f}$ in $\mathscr{E}_b(E_V)$. We endow $\mathscr{E}_b(E_V)$ with the topology of uniform convergence of the functions and all their derivatives on each bounded subset of E_V and $\mathscr{E}_{ub}(E)$ with the locally convex inductive limit topology of these spaces.

Now we recall some definitions in Nachbin–Dineen [13]. Let E be a normed space such that its strong dual E' has the approximation property (a.p. for short). We consider the completed topological tensor product $E'\pi^{\hat{\otimes} n}$ of E', n times and $L({}^nE) = L({}^nE; \mathbf{C})$ the space of the n-linear continuous functions on E, with its usual norm. We have a continuous injection:

$$E'\pi^{\otimes n} \xrightarrow{\chi} L({}^nE),$$

$$(\varphi_1 \otimes \cdots \otimes \varphi_n) \mapsto \varphi_1 \times \cdots \times \varphi_n,$$

where

$$(\varphi_1 \times \cdots \times \varphi_n)(x_1, \cdots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n),$$

which admits an injective continuation:

$$E'\pi^{\hat{\otimes} n} \xrightarrow{\tilde{\chi}} L({}^nE).$$

We define $\mathscr{P}_N({}^nE) = \mathscr{P}_N({}^nE; \mathbf{C})$ (the nuclear n-homogeneous polynomials) as the subspace of $E'\pi^{\hat{\otimes} n}$ made of those elements which are symmetric functions when considered via $\tilde{\chi}$ in $L({}^nE)$. $\mathscr{P}_N({}^nE)$ is equipped with the norm induced by $E'\pi^{\hat{\otimes} n}$ that is called the nuclear norm $\| \ \|_N$.

For the convenience of the sequel we are going to slightly reformulate this nuclear norm. Let P be a nuclear n-homogeneous polynomial. Then

$$(1) \qquad P = \sum_{j=0}^{\infty} \phi_{1_j} \otimes \cdots \otimes \phi_{n_j}, \qquad \text{where } \phi_{i_j} \in E' \quad (\text{Schaefer [14], p. 94}),$$

$$\|P\| = \inf_{\substack{\text{over all the} \\ \text{representations of} \\ P \text{ of the type (1)}}} \left\{ \sum_{j=0}^{\infty} \|\phi_{1_j}\|_{E'} \cdots \|\phi_{n_j}\|_{E'} \right\},$$

where

$$\|\phi_{i_j}\|_{E'} = \sup\{|\phi_{i_j}(x)|; \|x\| \leqq 1\} \qquad (\text{Schaefer [14], p. 93}).$$

We have the following:

LEMMA 1.1. *$\|P\|_N = \|P\|_{\Gamma_{l_1} B'^{\otimes n}}$, where $\| \ \|_{\Gamma_{l_1} B'^{\otimes n}}$ is the gauge of $\Gamma_{l_1} B'^{\otimes n}$ (B' is the closed unit ball of E').*

PROOF. First we recall that

$$\Gamma_{l_1} B'^{\otimes n} = \left\{ \sum_{i=0}^{\infty} \lambda_i T_{1_i} \otimes \cdots \otimes T_{n_i}, \text{ where } \sum_{i=0}^{\infty} |\lambda_i| \leqq 1 \text{ and } T_{i_j} \in B' \right\}.$$

If $\|P\|_{\Gamma_{l_1} B'^{\otimes n}} \leqq \mu$, then for every $\varepsilon > 0$, we may write

$$P = (\mu + \varepsilon) \sum_{i=0}^{\infty} \lambda_i T_{1_i} \otimes \cdots \otimes T_{n_i},$$

where

$$\sum_{i=0}^{\infty} |\lambda_i| \leqq 1, \qquad T_{i_j} \in B',$$

hence $\|P\|_N \leqq \mu + \varepsilon$, and then

$$\|P\|_N \leqq \|P\|_{\Gamma_{l_1} B'^{\otimes n}}.$$

Now, if $\|P\|_N \leqq \mu$, then for every $\varepsilon > 0$, $\|P\|_N \leqq \mu + \varepsilon$, hence we may write

$$P = \sum_{j=0}^{\infty} \phi_{1_j} \otimes \cdots \otimes \phi_{n_j}, \quad \text{where } \sum_{j=0}^{\infty} \|\phi_{1_j}\|_{E'} \cdots \|\phi_{n_j}\|_{E'} \leqq \mu + \varepsilon.$$

If $\mu_j = \|\phi_{1_j}\|_{E'} \cdots \|\phi_{n_j}\|_{E'}$, then $\Sigma_{j=0}^{\infty} |\mu_j| \leqq \mu + \varepsilon$ and

$$P = \sum_{i=0}^{\infty} \mu_j \frac{\phi_{1_j}}{\|\phi_{1_j}\|_{E'}} \otimes \cdots \otimes \frac{\phi_{n_j}}{\|\phi_{n_j}\|_{E'}}$$

hence $\|P\|_{\Gamma_{l_1} B'^{\otimes n}} \leqq \mu + \varepsilon$, and we have $\|P\|_{\Gamma_{l_1} B'^{\otimes n}} \leqq \|P\|_N$. ∎

Let E be a real normed space such that E' has the a.p. We denote by $\mathscr{E}_{\mathrm{Nb}}(E)$ (the infinitely nuclearly differentiable functions of bounded type on E) the subspace of all infinitely differentiable functions $f: E \to \mathbf{C}$, such that

(a) $\hat{d}^n f$ maps E into $\mathscr{P}_{\mathrm{N}}({}^n E)$, $\forall n \in \mathbf{N}$,

(b) $\hat{d}^n f : E \to \mathscr{P}_{\mathrm{N}}({}^n E)$ is differentiable and bounded on bounded subsets of E, $\forall n \in \mathbf{N}$.

The topology of $\mathscr{E}_{\mathrm{Nb}}(E)$ is the one generated by the following countable system of seminorms:

$$q_{m,n}(f) = \sup\{\|\hat{d}^i f(x)\|_N \,;\, 0 \leqq i \leqq n, \|x\| \leqq m\}, \qquad n, m = 0, 1, \cdots,$$

for every f in $\mathscr{E}_{\mathrm{Nb}}(E)$.

We denote by $\mathscr{E}_{\mathrm{Nbc}}(E)$ (the infinitely differentiable functions of bounded-compact type on E) the closure in $\mathscr{E}_{\mathrm{Nb}}(E)$ of the vector space generated by all the functions of the form $\phi^{\otimes n}$, $\phi \in E'$, $n \in \mathbf{N}$ (i.e., the continuous polynomials of finite type on E).

A counterexample of Abuabara [1] shows that $\mathscr{E}_{\mathrm{Nbc}}(E) \neq \mathscr{E}_{\mathrm{Nb}}(E)$, in general.

If E *is a real bornological vector space* (*b.v.s. for short*) *separated by its dual* E^*, we denote by $\mathscr{E}(E)$ the space of all infinitely differentiable functions on E, endowed with the topology of uniform convergence of the functions and their derivatives on the strictly compact subsets of E.

II. Structures of the spaces $\mathscr{E}_{\mathrm{ub}}(E)$ and $\mathscr{E}(E)$

We recall that if E is a real nuclear l.c.s., there is a base of 0-neighbourhoods (V_ι) in E such that the spaces E_{V_ι} are separable pre-Hilbert spaces and E is the projective limit of E_{V_ι} (Schaefer [14], p. 102).

THEOREM 2.1. *If E is a real nuclear l.c.s., then one has algebraically and topologically*

$$\mathscr{E}_{\mathrm{ub}}(E) = \textit{inductive limit of } \mathscr{E}_{\mathrm{Nbc}}(E_V),$$

when V ranges over a base of 0-neighbourhoods in E such that E_V is a separable pre-Hilbert space.

Now let E be a real b.v.s. and let $\mathscr{B}(E)$ be the set of all bounded closed convex balanced subsets of E. We recall that if E is a real nuclear b.v.s., then there is a bornological representation $E =$ inductive limit of E_B, where $B \in \mathscr{B}(E)$ and the spaces E_B are separable Hilbert spaces (Hogbe–Nlend [12]).

THEOREM 2.2. *If E is a real nuclear b.v.s., then algebraically and topologically*

$$\mathscr{E}(E) = \text{projective limit of } \mathscr{E}_{\mathrm{Nbc}}(E_B),$$

where $B \in \mathscr{B}(E)$ are such that the spaces E_B are separable Hilbert spaces and E is the inductive limit of E_B.

For the proofs of these theorems we use the following lemmas:

LEMMA 2.3. *If E_1 and E_2 are two real normed spaces such that E_1' has the a.p., with a nuclear linear mapping i from E_1 into E_2 and if f is in $\mathscr{E}_b(E_2)$, then $f \circ i$ is in $\mathscr{E}_{\mathrm{Nb}}(E_1)$. Moreover, the mapping*

$$\psi : \mathscr{E}_{\mathrm{b}}(E_2) \to \mathscr{E}_{\mathrm{Nb}}(E_1)$$

$$f \to f \circ i$$

is continuous.

PROOF. Since i is a nuclear map, for every $x \in E_1$,

$$i(x) = \sum_{n=1}^{\infty} \lambda_n x_n'(x) y_n,$$

with

$$\sum_{n=1}^{\infty} |\lambda_n| \leqq 1, \quad \|x_n'\|_{E_1'} \leqq 1, \quad \|y_n\|_{E_2} \leqq 1,$$

for each $n = 1, 2, \cdots$. Hence for n in $\mathbf{N}$,

$$(f \circ i)^{(n)}(x) h_1 \cdots h_n = \sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} x_{q_1}'(h_1) \cdots x_{q_n}'(h_n) f^{(n)}(ix) y_{q_1} \cdots y_{q_n}$$

and

$$\hat{d}^n (f \circ i)(x) = \sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} f^{(n)}(ix) y_{q_1} \cdots y_{q_n} x_{q_1}' \otimes \cdots \otimes x_{q_n}'. \tag{1}$$

Since $\sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} \leqq (\sum_{i=0}^{\infty} |\lambda_i|)^n \leqq 1$, and $f \in \mathscr{E}_{\mathrm{b}}(E_2)$, we have that $\hat{d}^n (f \circ i)(x) \in \mathscr{P}_{\mathrm{N}}({}^nE)$. Furthermore, the image through $\hat{d}^n (f \circ i)$ of a bounded subset in E is a bounded set in $\mathscr{P}_{\mathrm{N}}({}^nE)$, because $f \in \mathscr{E}_{\mathrm{b}}(E_2)$ and $\|y_{q_i}\|_{E_2} \leqq 1$. Now, we must prove that the map

$$\hat{d}^n (f \circ i) : E_1 \to \mathscr{P}_{\mathrm{N}}({}^nE_1)$$

is differentiable. Since

$$(f \circ i)^{(n)}(x + h) = \sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} f^{(n)}(i(x + h)) y_{q_1} \cdots y_{q_n} x_{q_1}' \times \cdots \times x_{q_n}'$$

and

$$f^{(n)}(ix+ih)=f^{(n)}(ix)+f^{(n+1)}(ix)ih+r_{ix}(ih),$$

where $r_{ix}(ih)$ is a remainder, we have that

$$\hat{d}^n(f\circ i)(x+h)-\hat{d}^n(f\circ i)(x)=\sum_{q_k}\lambda_{q_1}\cdots\lambda_{q_n}f^{(n+1)}(ix)ihy_{q_1}\cdots y_{q_n}x'_{q_1}\otimes\cdots\otimes x'_{q_n}$$

$$+\sum_{q_k}\lambda_{q_1}\cdots\lambda_{q_n}r_{ix}(ih)y_{q_1}\cdots y_{q_n}x'_{q_1}\otimes\cdots\otimes x'_{q_n}.$$

The first term of the above sum, considered as a function of h, is linear bounded. For the second term, we have

$$r_{ix}(ih)\in\tfrac{1}{2}\bar{\Gamma}\{f^{(n+2)}(ix)+ith(ih)^2\}_{0\leqq t\leqq 1},$$

where the closed convex balanced hull $\bar{\Gamma}$ is taken in the Banach space $L^n(E_2)$. (See Colombeau [4].) Then this term is contained in cte.$(\|h\|_{E_1})^2\cdot\Gamma_{l_1}(B_1'^{\otimes n})$, where B_1' is the closed unit ball in E_1'.

From Lemma 1.1, this proves the differentiability of the map $\hat{d}^n(f\circ i)$ from E_1 into $\mathcal{P}_N(^nE_1)$. Then $f\circ i$ is in $\mathcal{E}_{Nb}(E_1)$.

Since $\mathcal{E}_b(E_2)$ and $\mathcal{E}_{Nb}(E_1)$ are metrizable spaces, (1) gives also that the mapping

$$\mathcal{E}_b(E_2)\xrightarrow{\ \psi\ }\mathcal{E}_{Nb}(E_1)$$

$$f\longrightarrow f\circ i$$

is continuous. ■

REMARK. When we consider complex valued functions, we assume that all Nachbin algebras (conditions 5.2.4 in Colombeau [4]) in the following results are invariant under complex conjugation.

LEMMA 2.4 (Approximation). *Let E_1 and E_2 be two real normed spaces such that E_1' has the a.p., with a nuclear mapping i from E_1 into E_2, that is,*

$$i(x)=\sum_{n=1}^{\infty}\lambda_n x_n'(x)y_n,$$

with

$$\sum_{n=1}^{\infty}|\lambda_n|\leqq 1,\qquad \lambda_n\in\mathbf{C},$$

where $x'_n \in B'_1$, the unit ball of E'_1, $y_n \in B_2$, the unit ball of E_2 ($n = 1, 2, \cdots$), and $x \in E_1$.

For r in $\mathbf{N}$, let

$$i_r(x) = \sum_{n=1}^{r} \lambda_n x'_n(x) y_n,$$

for x in E_1, where λ_n, x'_n and y_n are as above in the representation of i.

If Y is a Nachbin subalgebra of $\mathscr{E}_b(E_2)$, if $\mathscr{V}$ is a 0-neighbourhood in $\mathscr{E}_{Nb}(E_1)$ and if f is an element of $\mathscr{E}_b(E_2)$, then for r large enough, there is an element ψ in Y, such that $f \circ i - \psi \circ i_r$ is in $\mathscr{V}$.

PROOF. We denote by B_1, B'_1 and B_2 the closed unit ball in E_1, E'_1 and E_2 respectively. Let

$$\mathscr{V} = \{\varphi \in \mathscr{E}_{Nb}(E_1) \text{ such that } \varphi^{(n)}(x) \in \tilde{\chi}(\nu \Gamma_{l_1}(B'^{\otimes n})), \text{ if } 0 \leqq n \leqq m \text{ and } x \in \mu B_1\},$$

for some $m \in \mathbf{N}$, μ and $\nu > 0$, be a 0-neighbourhood in $\mathscr{E}_{Nb}(E_1)$.

For r in $\mathbf{N}$, let us denote by $E_{2,r}$ the vector subspace of E_2 spanned by the vectors $y_1, \cdots, y_r$. We first prove that for r large enough, we have

$$f \circ i - f \circ i_r \in \tfrac{1}{2}\mathscr{V}. \tag{1}$$

Like in Lemma 2.3, for n in $\mathbf{N}$,

$$(f \circ i)^{(n)}(x) = \sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} f^{(n)}(ix) y_{q_1} \cdots y_{q_n} x'_{q_1} \times \cdots \times x'_{q_n}$$

and

$$(f \circ i_r)^{(n)}(x) = \sum_{q_\iota \leqq r} \lambda_{q_1} \cdots \lambda_{q_n} f^{(n)}(i_r x) y_{q_1} \cdots y_{q_n} x'_{q_1} \times \cdots \times x'_{q_n}.$$

In the difference $(f \circ i)^{(n)}(x) - (f \circ i_r)^{(n)}(x)$ there are two types of terms:

(I) A finite number of terms (with $q_\iota \leqq r$):

$$\sum_{q_\iota \leqq r} \lambda_{q_1} \cdots \lambda_{q_n} (f^{(n)}(ix) - f^{(n)}(i_r x)) y_{q_1} \cdots y_{q_n} x'_{q_1} \times \cdots \times x'_{q_n}.$$

Since i_r converges uniformly to i on μB_1, if $r \to \infty$, $i_r(\mu B_1) \subset \mu B_2$ and $f^{(n)}$ is uniformly continuous on μB_2, we have that

$$|(f^{(n)}(ix) - f^{(n)}(i_r x)) y_{q_1} \cdots y_{q_n}| \to 0, \quad \text{when } r \to \infty \quad \text{and} \quad x \in \mu B_1 \quad (0 \leqq n \leqq m).$$

(II) The infinite sum:

$$\sum_{q_k} \lambda_{q_1} \cdots \lambda_{q_n} f^{(n)}(ix) y_{q_1} \cdots y_{q_n} x'_{q_1} \times \cdots \times x_{q_n}$$

with at least one of the q_i's larger than r.

Let

$$d_r = \sum_{q_k} |\lambda_{q_1}| \cdots |\lambda_{q_n}| = \left(\sum_{q=1}^{\infty} |\lambda_q| \right)^n - \left(\sum_{q=1}^{r} |\lambda_q| \right)^n$$

with at least one of the q_i's larger than r. Then $d_r \to 0$ when $r \to \infty$.

Now, since $\{f^{(n)}(y)\}_{y \in \mu B_2}$ is a bounded subset of $L(^nE_2)$ and $i(\mu B_1) \subset \mu B_2$, $y_{q_i} \in B_2$, $x'_{q_i} \in B'_1$, we have (1) from (I) and (II), for r large enough.

Now we are going to prove that for any fixed given r large enough we have

$$\exists \psi \text{ in } Y, \text{such that } f \circ i_r - \psi \circ i_r \in \tfrac{1}{2}\mathcal{V}. \tag{2}$$

We set $f_r = f/E_{2,r}$ and then $f \circ i_r = f_r \circ i_r$.

In $\mathscr{E}(E_{2,r})$, let us apply Nachbin's approximation theorem: given $\varepsilon > 0$, there is φ_ε in $Y/E_{2,r}$, such that for every x in $\mu B_2 \cap E_{2,r}$, $0 \leqq n \leqq m$, then

$$\| \hat{d}^n f_r(x) - \hat{d}^n \varphi_\varepsilon(x) \|_{\mathscr{P}(^nE_{2,r})} \leqq \varepsilon. \tag{3}$$

If $\xi_j \in E_1$, $1 \leqq j \leqq n$, we have

$$(f_r \circ i_r)^{(n)}(x)\xi_1 \cdots \xi_n - (\varphi_\varepsilon \circ i_r)^{(n)}(x)\xi_1 \cdots \xi_n$$
$$= f_r^{(n)}(i_r x) i_r\xi_1 \cdots i_r\xi_n - \varphi_\varepsilon^{(n)}(i_r x) i_r\xi_1 \cdots i_r\xi_n.$$

Since in finite dimension the nuclear norm on $\mathscr{P}(^nE_{2,r})$ is equivalent to the usual norm, (3) implies that

$$f_r^{(n)}(i_r x) - \varphi_\varepsilon^{(n)}(i_r x) \in \tilde{\chi}(\varepsilon \Gamma_{l_1}(B_3'^{\otimes n})),$$

if r is large enough and B'_3 is the closed unit ball of $(E_{2,r})'$.

Therefore we may write

$$f_r^{(n)}(i_r x) - \varphi_\varepsilon^{(n)}(i_r x) = \varepsilon \sum_{q=1}^{\infty} \lambda_q T_q^1 \times \cdots \times T_q^n$$

with $\Sigma_{q=1} |\lambda_q| \leqq 1$ and T_q^n is in B'_3.

Now, if ξ_j is in E_1, $1 \leqq j \leqq n$, it follows that

$$(f_r \circ i_r)^{(n)}(x)\xi_1 \cdots \xi_n - (\varphi_\varepsilon \circ i_r)^{(n)}(x)\xi_1 \cdots \xi_n$$

$$= \varepsilon \sum_q \lambda_q (T_q^1 \times \cdots \times T_q^n)(i_r\xi_1 \cdots i_r\xi_n)$$

and thus

$$(f_r \circ i_r)^{(n)}(x) - (\varphi_\varepsilon \circ i_r)^{(n)}(x) = \varepsilon \sum_q \lambda_q (T_q^1 \circ i_r \times \cdots \times T_q^n \circ i_r),$$

with $T'_q \circ i_r$ in B'_1, $1 \leqq j \leqq n$.

Therefore

$$(f_r \circ i_r)^{(n)}(x) - (\varphi_\varepsilon \circ i_r)^{(n)}(x) \in \bar{\chi}(\varepsilon \Gamma_{l_1}(B_1'^{\otimes n}))$$

and this is true for every x in μB_l and $0 \leqq n \leqq m$.

Hence

$$f \circ i_r - \varphi_\varepsilon \circ i_r \text{ is in } \tfrac{1}{2}\mathcal{V}, \qquad \text{if } \varepsilon = \nu/2$$

(i.e., for r large enough).

Then we have (2) with $\psi \circ i_r = \varphi_\varepsilon \circ i_r$. From (1) and (2), $f \circ i - \psi \circ i_r$ is in $\mathcal{V}$. ■

As an immediate corollary of the above two lemmas we obtain:

LEMMA 2.5. *Let E_1 and E_2 be two real normed spaces such that E'_1 has the a.p., with a linear nuclear mapping i from E_1 into E_2, then if f is in $\mathscr{E}_b(E_2)$, $f \circ i$ is in $\mathscr{E}_{Nbc}(E_1)$ and the mapping*

$$\psi : \mathscr{E}_b(E_1) \to \mathscr{E}_{Nbc}(E_1)$$

$$f \mapsto f \circ i$$

is continuous.

PROOF. From Lemma 2.3, it suffices to prove that $f \circ i$ may be approximated in $\mathscr{E}_{Nb}(E_1)$ by finite type continuous polynomials on E_1. This follows from Lemma 2.4, if we take Y as the set of the continuous homogeneous polynomials on E_2. The continuity of ψ follows from Lemma 2.3. ■

PROOF OF THEOREM 2.1. If f is in $\mathscr{E}_{ub}(E)$, there are a 0-neighbourhood V_2 in E such that E_{V_2} is a separable pre-Hilbert space and $\tilde{f}$ in $\mathscr{E}_b(E_{V_2})$ such that $f = \tilde{f} \circ s_{V_2}$.

Since E is a nuclear l.c.s., there is a 0-neighbourhood V_1 in E such that the canonical map $i : E_{V_1} \to E_{V_2}$ is a nuclear map. From Lemma 2.5, $\tilde{f} \circ i$ is in $\mathscr{E}_{Nbc}(E_{V_1})$ and hence f is in the inductive limit of $\mathscr{E}_{Nbc}(E_V)$.

Conversely, if f/E_V is in $\mathscr{E}_{Nbc}(E_V)$, for some 0-neighbourhood V in E, such that E_V is a pre-Hilbert space, then f is in $\mathscr{E}_{ub}(E)$, trivially. Hence the algebraic equality.

The topological equality follows also by Lemma 2.5. ■

REMARK 1. This result is the C^∞-analogue of theorem 3.9 in Colombeau–Matos [8] for holomorphic functions.

REMARK 2. Note that if E is a DFN-space, $\mathscr{E}(E) = \mathscr{E}_{ub}(E)$ algebraically and topologically, so all structures coincide in this case. (See Colombeau–Mujica [9].)

PROOF OF THEOREM 2.2. Let B_1 be in $\mathcal{B}(E)$ such that E_{B_1} is a separable Hilbert space. Since E is a nuclear b.v.s., there is B_2 in $\mathcal{B}(E)$, $B_1 \subset B_2$ such that the inclusion mapping $i_1 : E_{B_1} \to E_{B_2}$ is nuclear. Also there is B_3 in $\mathcal{B}(E)$, $B_2 \subset B_3$, such that the inclusion mapping $i_2 : E_{B_2} \to E_{B_3}$ is nuclear. Hence mB_2 is relatively compact in E_{B_3}, for every $m \in \mathbf{N}$.

Thus if f is in $\mathcal{E}(E)$, $f/E_{B_2} = (f/E_{B_3}) \circ i_2$ is in $\mathcal{E}_b(E_{B_2})$.

By Lemma 2.5, $f/E_{B_1} = (f/E_{B_2}) \circ i_1$ is in $\mathcal{E}_{Nbc}(E_{B_1})$. This implies that f is in the projective limit of $\mathcal{E}_{Nbc}(E_B)$.

From the trivial inclusion: projective limit of $\mathcal{E}_{Nbc}(E_B) \subset \mathcal{E}(E)$ and Lemma 2.5, we have the algebraic and topological equality. ∎

REMARK. This result is the C^∞-analogue of theorem 3.6 in Colombeau–Matos [8], for holomorphic functions.

III. Density results

THEOREM 3.1. *Let E be a real nuclear l.c.s. and Y be a subalgebra of $\mathcal{E}_{ub}(E)$ such that there is a base $(V_\iota)_\iota$ of convex balanced pre-Hilbertian 0-neighbourhoods in E satisfying:*

(1) *For every i,*

$$\tilde{Y}^{V_\iota} = \{f \in \mathcal{E}_b(E_{V_\iota}) \text{ such that } f \circ s_{V_\iota} \in Y\}$$

is a Nachbin subalgebra of $\mathcal{E}_b(E_{V_\iota})$.

(2) *Given V_2 in $(V_\iota)_\iota$, there is V_1 in $(V_\iota)_\iota$ such that the canonical mapping $i : E_{V_1} \to E_{V_2}$ is nuclear, i.e.,*

$$i(x) = \sum_{n=1}^{\infty} \lambda_n x'_n(x) y_n,$$

with $\sum_{n=1}^{+\infty} |\lambda_n| \leqq 1$, x'_n in B'_1, the closed unit ball in E'_{V_1}, y_n in B_2, the closed unit ball in E_{V_2} $(n = 1, 2, \cdots)$, and x in E_{V_1}.

If we write for $r \in \mathbf{N}$

$$i_r(x) = \sum_{n=1}^{r} \lambda_n x'_n(x) y_n,$$

with λ_n, x'_n, y_n as above and x in E_{V_1}, we assume furthermore that for r large enough, the set

$$\{f \circ i_r, \text{ for } f \in \tilde{Y}^{V_2}\}$$

is contained in $\tilde{Y}^{V_1}$.

Then Y is dense in $\mathcal{E}_{ub}(E)$.

COROLLARY 3.2. *If E is a real nuclear l.c.s., then $\mathcal{P}_f(E)$ (the finite type continuous polynomials in E) is dense in $\mathcal{E}_{ub}(E)$.*

COROLLARY 3.3. *If E is a real nuclear l.c.s., then the set*

$$\left\{\sum_{\text{finite}} c_j e^{i\varphi_j},\ c_j \in \mathbf{C},\ \varphi_j \in E'\right\}$$

is dense in $\mathcal{E}_{ub}(E)$.

Let E = inductive limit of E_B, $B \in \mathcal{B}(E)$, be a real bornological vector space separated by its dual E^*. We recall that a subset Ω of E is said to be τE-open, if for all B in $\mathcal{B}(E)$, $\Omega \cap E_B$ is an open subset of E_B.

THEOREM 3.4. *Let E be a real nuclear b.v.s. and let $\mathcal{T}$ be a Hausdorff locally convex topology on E for which any bounded subset of E is $\mathcal{T}$-bounded. Let Ω be a τE-open subset. A subalgebra A in $\mathcal{E}(\Omega)$ is dense in $\mathcal{E}(\Omega)$ if and only if the following conditions are satisfied:*

(a) *A is a Nachbin subalgebra;*

(b) *for every B in $\mathcal{B}(E)$ such that E_B is a separable Hilbert space, every u in $(E_B)' \otimes (E_B)$, every open subset $\omega \subset \Omega \cap E_B$, with $u(\omega) \subset \Omega$, and every g in A, the composition mapping $g \circ (u/\omega)$ is in the closure of A/ω in $\mathcal{E}(\omega)$.*

REMARK. The above theorem improves theorem 5.2.1 of Colombeau [4].

PROOF OF THEOREM 3.1. Let f be in $\mathcal{E}_{ub}(E)$ and $\mathcal{V}$ be a 0-neighbourhood in $\mathcal{E}_{ub}(E)$. We desire to prove that there is ψ in Y such that $f - \psi \in \mathcal{V}$. By definition of $\mathcal{E}_{ub}(E)$, there are a 0-neighbourhood V_2 in E and a function $\tilde{f}$ in $\mathcal{E}_b(E_{V_2})$ such that $f = \tilde{f} \circ s_{V_2}$. Since E is a nuclear l.c.s., there is a 0-neighbourhood V_1 in E, such that the linear canonical map $i : E_{V_1} \to E_{V_2}$ is nuclear. By Theorem 2.1, there is a 0-neighbourhood $\mathcal{V}_1$ in $\mathcal{E}_{Nbc}(E_{V_1})$ such that if φ is in $\mathcal{V}_1$, then $\varphi \circ s_{V_1}$ is in $\mathcal{V}$.

Now, it suffices to show that there is ψ_1 in $\mathcal{E}_b(E_{V_1})$ such that $\psi_1 - \tilde{f} \circ i \in \mathcal{V}_1$ and $\psi_1 \circ s_{V_1}$ is in Y.

By Lemma 2.4 and (1) of the hypothesis, there is $\tilde{\psi}$ in $\tilde{Y}^{V_2}$ such that $\tilde{f} \circ i - \tilde{\psi} \circ i_r$ is in $\mathcal{V}_1$, for r large enough. Take $\psi_1 = \tilde{\psi} \circ i_r$ and then $\psi_1 \circ s_{V_1} = (\tilde{\psi} \circ i_r) \circ s_{V_1}$ is in Y, by (2) of the hypothesis. ■

PROOF OF THEOREM 3.4. The proof of the sufficiency in this theorem is in the proof of theorem 5.2.1 of Colombeau [4].

Now, let A in $\mathcal{E}(\Omega)$ be a dense subalgebra. It is classical that A is necessarily a Nachbin subalgebra. Next, let u, B and ω be as in condition (b) of the theorem and let g in A be given.

Since E_B is a Hilbert space, $(E_B)'$ has the bounded approximation property. then by Aron–Prolla [3], we have that $\mathcal{P}_f(E_B)/\omega$ is dense in $\mathcal{E}(\omega)$ and so $g \circ u/\omega$ belongs to the closure of $\mathcal{P}_f(E_B)/\omega$ in $\mathcal{E}(\omega)$.

Since the restriction mapping:

$$r : (E, \mathcal{T})' \to (E_\beta)'_{\tau_0}$$

has dense range (see Colombeau [4] 5.1.6), then $g \circ u/\omega$ belongs to the closure of $\mathcal{P}_f(E, \mathcal{T})/\omega$ in $\mathcal{E}(\omega)$.

On the other hand, by the density of A, every element of $\mathcal{P}_f(E, \mathcal{T})/\Omega$ belongs to the closure of A in $\mathcal{E}(\Omega)$. *A fortiori* every element of $\mathcal{P}_f(E, \mathcal{T})/\omega$ belongs to the closure of A/ω in $\mathcal{E}(\omega)$, which shows that A satisfies the condition (b).

IV. A version of the Paley–Wiener–Schwartz theorem

DEFINITION 4.1. Let E be a real nuclear l.c.s. The Fourier transform $\mathcal{F}$ from $\mathcal{E}'_{\text{ub}}(E)$ into $\mathcal{H}_S(E'_{\mathbf{C}})$ is defined by

$$(\mathcal{F}l)(\xi) = l(e^{i\xi}),$$

if $\xi \in E'_{\mathbf{C}}$, if l is in $\mathcal{E}'_{\text{ub}}(E)$, where $E'_{\mathbf{C}}$ denotes the complexification of the strong dual E' of E and $\mathcal{H}_S(E'_{\mathbf{C}})$ denotes the set of the S-holomorphic functions on $E'_{\mathbf{C}}$.

$\mathcal{F}$ is injective from Corollary 3.3.

DEFINITION 4.2. Let E be a real nuclear b.v.s. The Fourier transform $\mathcal{F}$ from $\mathcal{E}'(E)$ into $\mathcal{H}(E^*_{\mathbf{C}})$ is defined by

$$(\mathcal{F}l)(\xi) = l(e^{i\xi}),$$

if $\xi \in E^*_{\mathbf{C}}$ and l is in $\mathcal{E}'(E)$. We recall that E^* is endowed with its natural topology and $\mathcal{H}(E^*_{\mathbf{C}})$ denotes the set of the holomorphic functions in $E^*_{\mathbf{C}}$.

$\mathcal{F}$ is injective from theorem 5.2.6 in Colombeau [4].

Let E be a real nuclear b.v.s., E = inductive limit of E_B, such that $B \in \mathcal{B}(E)$ and E_B is a separable Hilbert space. Let us equip $\mathcal{E}'(E)$ with its equicontinuous bornology and its strong topology. We have by Theorem 2.2 that

(1) $$\mathcal{E}(E) = \text{projective limit of } \mathcal{E}_{\text{Nbc}}(E_B).$$

Now, since the restriction map

$$\mathcal{E}(E) \longrightarrow \mathcal{E}_{\text{Nbc}}(E_B)$$

$$\varphi \longrightarrow \varphi/E_B$$

for each B in $\mathscr{B}(E)$ has a dense range (it follows by the fact that the restriction map $E^* \to (E_B)'$ has a dense range, when E_B is a Hilbert space (see Colombeau [4] 7.2.3)), and if $E_{B_1} \subset E_{B_2}$, the restriction map from $\mathscr{E}_{\mathrm{Nbc}}(E_{B_2})$ into $\mathscr{E}_{\mathrm{Nbc}}(E_{B_1})$ has a dense range, hence its transpose from $\mathscr{E}'_{\mathrm{Nbc}}(E_{B_1})$ into $\mathscr{E}'_{\mathrm{Nbc}}(E_{B_2})$ is injective. Then algebraically one has, from (1), that

$$\mathscr{E}'(E) = \text{inductive limit of } \mathscr{E}'_{\mathrm{Nbc}}(E_B),$$

when B is in $\mathscr{B}(E)$ and E_B is a separable Hilbert space (and the equicontinuous subsets of $\mathscr{E}'(E)$ are the subsets contained and equicontinuous in some $\mathscr{E}'_{\mathrm{Nbc}}(E_B)$).

Thus, since $\mathscr{F}$ is injective, we have

$$\mathscr{F}\mathscr{E}'(E) = \text{inductive limit of } \mathscr{F}\mathscr{E}'_{\mathrm{Nbc}}(E_B)$$

when $B \in \mathscr{B}(E)$ is such that E_B is a separable Hilbert space. Now, by Abuabara [1], $\mathscr{F}\mathscr{E}'_{\mathrm{Nbc}}(E_B) = A_{\mathrm{eq}}((E'_B)_C)$ is the vector space of all f in $\mathscr{H}((E'_B)_C)$, such that:

(1) there are constants $c > 0$, $m > 0$ and $\nu \in \mathbf{N}$, such that

$$|f(\xi)| \leqq c(1 + \|\xi\|_B)^\nu \exp(m \|\mathrm{Im}\, \xi\|_B),$$

for every $\xi \in (E_B)'_C$, where $\mathrm{Im}\, \xi$ denotes the imaginary part of ξ.

(2) The family $(\xi_n)_n$ in Y' (where Y is the vector subspace of $\mathscr{E}_{\mathrm{Nbc}}(E_B)$ generated by all the mappings of the form $e^{i\varphi} : E_B \to \mathbf{C}$ where $\varphi \in E'_B$), defined by

$$g = \sum_{j=1}^{l} \alpha_j e^{i\varphi_j} \in Y \to \xi_n(g) = \sum_{j=1}^{l} \alpha_j f(\varphi_j \circ P_n)$$

is equicontinuous. ($P_n : E_B \to E_B$ are the projections on the first n vectors of a basis of E_B.)

Then

$$\mathscr{F}\mathscr{E}'(E) = \text{inductive limit of } A_{\mathrm{eq}}((E'_B)_C).$$

This remark gives the Paley–Wiener–Schwartz theorem proved in Ansemil–Colombeau [2] with another method.

Now, let E be a real nuclear l.c.s. and let E' be the dual of E with its equicontinuous bornology.

Definition 4.3. We denote by $\mathscr{A}$ the vector subspace of $\mathscr{H}_S(E'_C)$ made of the functions f such that, for every convex balanced 0-neighbourhood V in E, such that E_V is a separable pre-Hilbert space,

(1) $|f/(E'_{\mathring{V}})_{\mathrm{C}}(x)| \leqq c(1+\|x\|_{\mathring{V}})^{\nu} \exp(m\|\operatorname{Im} x\|_{\mathring{V}})$ for every x in $(E'_{\mathring{V}})_{\mathrm{C}}$ and some $c>0$, ν, $m>0$ (we recall that $(E'_{\mathring{V}})_{\mathrm{C}} = (E_V)'_{\mathrm{C}}$ and that $\mathring{V}$ denotes the polar set of V).

(2) The sequence $(\xi_n)_n$ in Y'_V (where $Y'_V \subset \mathscr{E}_b(E_V)$ is the subspace spanned by the functions of the form $e^{i\varphi}$, $\varphi \in (E_V)'$), defined by

$$g=\sum_{j=1}^{l} \alpha_j e^{i\varphi_j} \rightarrow \xi_n(g)=\sum_{j=1}^{l} \alpha_j f(\varphi_j \circ P_n)$$

(where $P_n : E_V \rightarrow E_V$ are the projections on the first n vectors of a basis of E_V), is equicontinuous.

We remark that from (1) of the latter definition, $\mathscr{A}$ is contained in $\operatorname{Exp}_S(E'_{\mathrm{C}})$ (see definition 6.4 in Colombeau–Matos [7]).

This space $\mathscr{A}$ is equipped with its bornology naturally derived from (1) and (2), that is, a family $(f_\iota)_{\iota\in I}$ is bounded iff for every V in (1) the constants c, ν, m are uniform in $i \in I$ and in (2) the family $(\xi_{n,\iota})_{\iota,n}$ is equicontinuous.

Let us equip the space $\mathscr{E}'_{\mathrm{ub}}(E)$ with its equicontinuous bornology.

THEOREM 4.4. *The Fourier transform $\mathscr{F}$ is a bornological isomorphism between $\mathscr{E}'_{\mathrm{ub}}(E)$ and $\mathscr{A}$.*

PROOF. Let V be a convex balanced 0-neighbourhood in E such that E_V is a separable pre-Hilbert space and let us denote by $s_V : E \rightarrow E_V$ the canonical surjective map. Let ${}^t s_V : (E_V)'_{\mathrm{C}} \rightarrow E'_{\mathrm{C}}$ be its transpose (${}^t s_V$ is injective).

We have the map:

$$\mathscr{E}_{\mathrm{Nbc}}(E_V) \xrightarrow{I_V} \mathscr{E}_{\mathrm{ub}}(E)$$
$$f \longrightarrow f \circ s_V$$

and its transpose ${}^t I_V$.

Let T be an element of $\mathscr{E}'_{\mathrm{ub}}(E)$ and ξ be in ${}^t s_V(E_V)'_{\mathrm{C}}$, that is, $\xi = {}^t s_V(\mu) = \mu \circ s_V$ for some μ in $(E_V)'_{\mathrm{C}}$. Thus

$$\widehat{{}^t I_V(T)}(\mu) = {}^t I_V(T)(e^{i\mu}) = (T \circ I_V)(e^{i\mu}) = T(e^{i(\mu\circ s_V)})$$
$$= T(e^{i\xi}) = \mathscr{F}T(\xi),$$

where $\wedge$ denotes the usual Fourier transform in $\mathscr{E}'_{\mathrm{Nbc}}(E_V)$.

Hence $\mathscr{F}T/{}^t s_V(E_V)'_{\mathrm{C}} = \widehat{{}^t I_V(T)}$ belongs to $A_{\mathrm{eq}}((E'_V)_{\mathrm{C}})$, by Abuabara [1]. Since it holds for every V, $\mathscr{F}T \in \mathscr{A}$.

Conversely, let U be in $\mathscr{A}$. Given a 0-neighbourhood V in E, such that E_V is a separable pre-Hilbert space, $U/((E)'_{\mathring{V}})_{\mathrm{C}}$ belongs to $A'_{\mathrm{eq}}(((E)'_{\mathring{V}})_C)$. Thus $U \circ {}^t s_V$ is in $A_{eq}((E_V)'_C)$. By Abuabara [1], there is an element T_V in $\mathscr{E}'_{Nbc}(E_V)$, such that $\widehat{T_V} = (U \circ {}^t s_V)$. Hence $T_V(e^{i\mu}) = U({}^t s_V(\mu))$, for every μ in $(E_V)'_C$.

If $V_1 \subset V_2$ are two 0-neighbourhoods in E,

$$ {}^t s_{V_2}(E'_{V_2})_C \subset {}^t s_{V_1}(E'_{V_1})_C $$

and we have a restriction mapping

$$ r : \mathscr{E}_{Nbc}(E_{V_2}) \longrightarrow \mathscr{E}_{Nbc}(E_{V_1}), $$

$$ \varphi \longrightarrow \varphi / E_{V_1}. $$

$T_{V_2}(\varphi) = T_{V_1}(r(\varphi))$, for every φ in $\mathscr{E}_{Nbc}(E_{V_2})$, since

$$ T_{V_2}(e^{i\xi}) = U({}^t s_{V_2}(\xi)), $$

and

$$ T_{V_1}(e^{ir(\xi)}) = U({}^t s_{V_1} \circ r(\xi)) = U({}^t s_{V_2}(\xi)) $$

and since $\{e^{i\xi}, \xi \in (E_{V_2})'_C\}$ is dense in $\mathscr{E}_{Nbc}(E_{V_2})$ (see Abuabara [1]).

Now we define T from $\mathscr{E}_{ub}(E)$ into $\mathbf{C}$ by $T(f) = T_V(f_V)$, if $f = f_V \circ s_V$, with $f_V \in \mathscr{E}_{Nbc}(E_V)$. Then $\mathscr{F}T = U$ and we have the algebraic equality between $\mathscr{F}(\mathscr{E}'_{ub}(E))$ and $\mathscr{A}$.

We remark that the Fourier transform from $\mathscr{E}'_{Nbc}(E_V)$ onto $A_{eq}((E_V)'_C)$ is a bounded mapping. This follows using Abuabara's proof:

Let $\{f_\alpha\}_{\alpha \in I}$ be a bounded subset of $\mathscr{E}'_{Nbc}(E_V)$. Then there are $c > 0$, m, $\nu \in \mathbf{N}$ such that

$$ |f_\alpha(g)| \leqq c \cdot \sup\{\|\hat{d}^k g(x)\|_N, k \leqq \nu, \|x\| \leqq m\}, $$

for every $\alpha \in I$. Now if $\xi = \varphi + i\psi$ is in $(E_V)'_C$, then for $k \leqq \nu$,

$$ \hat{d}^k(e^{i\xi})(x) = (i)^k e^{i\xi}(x) \cdot \xi^k. $$

Therefore,

$$ \sup\{\|\hat{d}^k(e^{i\xi})(x)\|_N, k \leqq \nu, \|x\| \leqq m\} \leqq (1 + \|\xi\|)^\nu e^{m\|\mathrm{Im}\,\xi\|} $$

and

$$ |\hat{f}_\alpha(\xi)| = |f_\alpha(e^{i\xi})| \leqq c(1 + \|\xi\|)^\nu e^{m\|\mathrm{Im}\,\xi\|}. $$

The equicontinuity of the family $(\xi_{n,\alpha})_{n,\alpha}$ defined in (2) in the definition of $A_{eq}((E'_V)_C)$ follows from the equicontinuity of $(f_\alpha)_{\alpha \in I}$. Then $\{\hat{f}_\alpha\}_{\alpha \in I}$ is a bounded subset of $A_{eq}((E'_V)_C)$.

Since $\mathscr{E}'_{\mathrm{Nbc}}(E_V)$ and $A_{\mathrm{eq}}((E'_V)_C)$ are complete b.v.s. and $\mathscr{E}'_{\mathrm{Nbc}}(E_V)$ has a countable basis, the closed graph theorem (Hogbe–Nlend [12], prop. 2, p. 44) gives that this Fourier transform is a bornological isomorphism.

The bornological isomorphism in Theorem 4.4 follows from this. ∎

REFERENCES

1. T. Abuabara, *A version of the Paley–Wiener–Schwartz theorem in infinite dimension*, in *Advances in Holomorphy* (J. A. Barroso, ed.), North-Holland Mathematics Studies, 34, 1979.

2. J. M. Ansemil and J. F. Colombeau, *The Paley–Wiener–Schwartz isomorphism in nuclear spaces*, Rev. Roum. Math. Pures Appl. **26** (1981), 169–181.

3. R. M. Aron and J. B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, J. Reine Angew. Math. **313** (1980), 195–216.

4. J. F. Colombeau, *Differential calculus and holomorphy*, in *Real and Complex Analysis in Locally Convex Spaces*, North-Holland Math. Studies, 64, 1982.

5. J. F. Colombeau, *A multiplication of distributions*, J. Math. Anal. Appl., in press.

6. J. F. Colombeau, *New generalized functions, multiplication of distributions, Physical applications, Contributions of J. Sebastião e Silva*, Port. Math., in press.

7. J. F. Colombeau and M. Matos, *Convolution equations in spaces of infinite dimensional entire functions*, Proc. K. Ned. Akad. Wet., Ser. A, **83** (4) (1980), 357.

8. J. F. Colombeau and M. Matos, *On some spaces of entire functions defined on infinite dimensional spaces*, Pac. J. Math. **92** (2) (1981).

9. J. F. Colombeau and J. Mujica, *Holomorphic and C^∞-functions of uniform bounded type*, in *Functional Analysis, Holomorphy and Approximation Theory* (J. A. Barroso, ed.), North-Holland Math. Studies **71**, 1982, pp. 179–200.

10. J. F. Colombeau and O. W. Paques, *Finite-difference partial differential equations in normed and locally convex spaces*, in *Functional Analysis, Holomorphy and Approximation Theory* (J. A. Barroso, ed.), North-Holland Math. Studies **71**, 1982, pp. 201–214.

11. C. P. Gupta, *Malgrange Theorem for nuclearly entire functions of bounded type on Banach spaces*, Thesis, University of Rochester, 1966, Notas de Matemática, 37 IMPA, Rio de Janeiro, 1968.

12. H. Hogbe-Nlend, *Théorie des Bornologies et Applications*, Lecture Notes in Math. **213**, Springer-Verlag, 1971.

13. L. Nachbin and S. Dineen, *Entire functions of exponential type bounded on the real axis and Fourier transforms of distributions with bounded support*, Isr. J. Math. **13** (1972), 321–326.

14. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, Berlin, 1971.

15. F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York–London, 1967.

U.E.R. DE MATHÉMATIQUES ET D'INFORMATIQUE
UNIVERSITÉ DE BORDEAUX 1
33.405 TALENCE, FRANCE

DEPARTAMENTO DE MATEMÁTICA
IMECC, UNICAMP
UNIVERSIDADE ESTADUAL DE CAMPINAS
13.100 CAMPINAS SP, BRASIL